

Currents algebra for an atom-molecule Bose-Einstein condensate model

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Abstract

I present an interconversion currents algebra for an atom-molecule Bose-Einstein condensate model and use it to get the quantum dynamics of the currents. For different choices of the Hamiltonian parameters I get different currents dynamics.

1 Introduction

Since the first experimental verification of the Bose-Einstein condensation (BEC) [1–3], occurred more than seven decades after its theoretical prediction [4, 5], a great effort in the theoretical and experimental viewpoint has been made in the study of this quantum many body physical phenomenon [6–13]. Looking for new applications an atom-molecule Bose-Einstein condensates was experimentally produced applying magnetic field near a Feshbach resonance in an atomic BEC [14, 15]. The magnetic field pulses produces a coherent interconversion dynamics of atoms and molecules. Another techniques used to produce an atom-molecule Bose-Einstein condensate are the photoassociation [16] and the two-photons Raman transition [17]. These experimental opened the possibility of the cold and ultracold quantum chemistry [17–32]. I am considering here an atom-molecule Bose-Einstein condensate model, used to study this coherent interconversion dynamics of atoms and homonuclear diatomic molecules [33, 34]. This model is integrable in the sense that it can be solved by the quantum inverse scattering method (QISM) [35–48]. In this work I will discuss the symmetries and the interconversion currents algebra of the model and use they to study the quantum dynamics of the currents. This method was applied in the study of the Josephson tunnelling phenomenon in a two-site Bose-Hubbard model to get the quantum dynamics of the tunnelling currents of atoms as a function of the parameters of the Hamiltonian [49, 50]. We can also apply this method to a model recently used to study the tunnelling between two atom-molecule BECs coupled by Josephson tunnelling [51–55]. It is also worth to note that we can consider the atom-molecule BEC as a two mode system, an atomic mode and the another one molecular. This bipartite system is entangled by the interconversion dynamic of atoms and molecules and we can study the quantum phase transition of the system using tools of the quantum information [34, 44]. The dynamics of interconversion of atoms and molecules as an open system, with particles losses was studied in [56]. We are considering a closed system with the total number of atoms conserved [49]. The dynamic of interconversion in an atom-molecule Bose-Einstein condensate is described by the Hamiltonian

$$\hat{H} = U_a \hat{N}_a^2 + U_b \hat{N}_b^2 + U_{ab} \hat{N}_a \hat{N}_b + \mu_a \hat{N}_a + \mu_b \hat{N}_b - \Omega(\hat{a}^\dagger \hat{a}^\dagger \hat{b} + \hat{b}^\dagger \hat{a} \hat{a}), \quad (1.1)$$

where \hat{a}^\dagger and \hat{a} are the creation and annihilation boson operators of an unbound atom while \hat{b}^\dagger and \hat{b} are the creation and annihilation boson operators of a molecule. The number boson operators $\hat{N}_a = \hat{a}^\dagger \hat{a}$ and $\hat{N}_b = \hat{b}^\dagger \hat{b}$ are the respective number of unbound atoms and number

of molecules operators. These bosons operators satisfies the canonical commutation relations

$$[\hat{a}, \hat{a}^\dagger] = [\hat{b}, \hat{b}^\dagger] = \hat{I}, \quad (1.2)$$

with the condition that the unbound atom and molecule boson operators commutes

$$[\hat{a}^\dagger, \hat{b}] = [\hat{a}, \hat{b}] = 0, \quad (1.3)$$

and

$$[\hat{N}_a, \hat{a}] = -\hat{a}, \quad [\hat{N}_a, \hat{a}^\dagger] = +\hat{a}^\dagger, \quad (1.4)$$

$$[\hat{N}_b, \hat{b}] = -\hat{b}, \quad [\hat{N}_b, \hat{b}^\dagger] = +\hat{b}^\dagger, \quad (1.5)$$

where \hat{I} is the identity operator.

The parameter U_a is the atom-atom interaction, U_{ab} is the atom-molecule interaction and U_b is the molecule-molecule interaction. The strength of these interactions are proportional to the scattering length and usually is enough to consider only the low energy s -wave scattering length. The parameter μ_a is the external potential for the unbound atoms and μ_b is the external potential for the molecules. The parameter Ω is the amplitude for interconversion of atoms and molecules. In the limit $U_a = U_{ab} = U_b = 0$, the Hamiltonian (1.1) has been studied using a variety of methods [57–61]. However in the experimental context, the s -wave scattering interactions play a significant role. It will be seen below that for the general model (1.1) the inclusion of these scattering terms has a non-trivial consequence. The s -wave scattering length for the atom-atom interaction U_a can be determined precisely by photoassociation spectroscopy of one and two photons [29,71]. We mention that generally the values for the molecule-molecule U_b and the atom-molecule U_{ab} interactions are unknown [17], although some estimates exist in the case of ^{85}Rb [72]. For an up to dated discussion of theoretical and experimental data see [71].

The paper is organized as follows. In the section 2, I present the symmetries of the Hamiltonian. In the section 3, I present the interconversion currents algebra and the Casimir operators. In the section 4, I calculate the quantum dynamics of the currents. In the section 5, I summarize the results.

2 Symmetries

The Hamiltonian (1.1) is invariant under the global $U(1)$ gauge transformation $\hat{a} \rightarrow e^{i\alpha}\hat{a}$, $\hat{b} \rightarrow e^{2i\alpha}\hat{b}$, where α is an arbitrary c -number and $\hat{a}^\dagger \rightarrow e^{-i\alpha}\hat{a}^\dagger$, $\hat{b}^\dagger \rightarrow e^{-2i\alpha}\hat{b}^\dagger$. The global $U(1)$

gauge invariance is associated with the conservation of the total number of atoms (unbound atoms plus bound atoms) $\hat{N} = \hat{N}_a + 2\hat{N}_b$. The Hamiltonian (1.1) is not invariant under the exchange of atoms and molecules because the interconversion dynamics is not symmetric under this exchange.

The Hamiltonian (1.1) is invariant under the \mathbb{Z}_2 mirror transformation of the unbound atom operators $\hat{a} \rightarrow -\hat{a}, \hat{a}^\dagger \rightarrow -\hat{a}^\dagger$, but it is not invariant under the \mathbb{Z}_2 mirror transformation of the molecule operators $\hat{b} \rightarrow -\hat{b}, \hat{b}^\dagger \rightarrow -\hat{b}^\dagger$. The \mathbb{Z}_2 symmetry is associated with the parity of the wave function by the relation

$$\hat{P}_a |\Psi\rangle = (-1)^{N_a} |\Psi\rangle, \quad (2.6)$$

where \hat{P}_a is the parity operator acting only on \hat{a} and \hat{a}^\dagger . Because this symmetry we have $[\hat{H}, \hat{P}_a] = 0$. When we consider the parity operator \hat{P}_b acting only on the molecule operators \hat{b} and \hat{b}^\dagger , the parity of the wave function is not well defined and we have $[\hat{H}, \hat{P}_b] \neq 0$.

If N_a is even, N is even and the dimension of the state space is $D = N/2 + 1$. For N_a even we can write a general state of the Hamiltonian (1.1) as

$$|\Psi\rangle = \sum_{n_b=0}^{\frac{N}{2}} C_{N-2n_b, n_b} \frac{(\hat{a}^\dagger)^{N-2n_b}}{\sqrt{(N-2n_b)!}} \frac{(\hat{b}^\dagger)^{n_b}}{\sqrt{n_b!}} |0, 0\rangle, \quad (2.7)$$

and

$$\hat{P}_a |\Psi\rangle = |\Psi\rangle. \quad (2.8)$$

If N_a is odd, N is odd and the dimension of the state space is $D = (N+1)/2$. For N_a odd we can write a general state of the Hamiltonian (1.1) as

$$|\Psi\rangle = \sum_{n_b=0}^{\frac{N-1}{2}} C_{N-2n_b, n_b} \frac{(\hat{a}^\dagger)^{N-2n_b}}{\sqrt{(N-2n_b)!}} \frac{(\hat{b}^\dagger)^{n_b}}{\sqrt{n_b!}} |0, 0\rangle, \quad (2.9)$$

and

$$\hat{P}_a |\Psi\rangle = -|\Psi\rangle. \quad (2.10)$$

The symmetries of the Hamiltonian (1.1) imply degeneracy. For the conservancy of \hat{N} we have that all wave function of the Hamiltonian (1.1) are degenerated eigenfunctions of \hat{N} with the same eigenvalue N . For the parity operator \hat{P}_a all wave function of the Hamiltonian (1.1) are even or odd depending if N is even or odd. All wave functions are degenerated

eigenfunctions of \hat{P}_a with the same eigenvalue $\lambda = +1$ if N is even or they are degenerated eigenfunctions of \hat{P}_a with the same eigenvalue $\lambda = -1$ if N is odd.

3 Interconversion Currents Algebra

The quantum dynamics of any operator \hat{O} in the Heisenberg picture is determined by the Heisenberg equation of motion

$$\frac{d\hat{O}}{dt} = \frac{i}{\hbar}[\hat{H}, \hat{O}]. \quad (3.11)$$

The boson operator total number of atoms, $\hat{N} = \hat{N}_a + 2\hat{N}_b$, is a conserved quantity, $[\hat{H}, \hat{N}] = 0$, and it is commutable compatible operator (CCO) with the number of unbound atoms and number of molecules boson operators, $[\hat{N}, \hat{N}_a] = [\hat{N}, \hat{N}_b] = [\hat{N}_a, \hat{N}_b] = 0$. The boson operators number of unbound atoms and the number of molecules don't commute with the Hamiltonian and their time evolution is dictated by the interconversion current operator,

$$\hat{\mathcal{J}} = \frac{1}{4i}(\hat{a}^\dagger \hat{a}^\dagger \hat{b} - \hat{b}^\dagger \hat{a} \hat{a}), \quad (3.12)$$

in coherent opposite phases because of the conservancy of \hat{N} , with

$$[\hat{H}, \hat{N}_a] = +8i\Omega\hat{\mathcal{J}}, \quad [\hat{H}, \hat{N}_b] = -4i\Omega\hat{\mathcal{J}}, \quad (3.13)$$

and

$$\frac{d\hat{N}_a}{dt} = -8\frac{\Omega}{\hbar}\hat{\mathcal{J}}, \quad (3.14)$$

$$\frac{d\hat{N}_b}{dt} = +4\frac{\Omega}{\hbar}\hat{\mathcal{J}}. \quad (3.15)$$

Integrating the Eqs. (3.14) and (3.15) we get

$$\hat{N}_a(t) = \hat{N}_a(0) - 8\frac{\Omega}{\hbar} \int_0^t \hat{\mathcal{J}}(\tau) d\tau, \quad (3.16)$$

$$\hat{N}_b(t) = \hat{N}_b(0) + 4\frac{\Omega}{\hbar} \int_0^t \hat{\mathcal{J}}(\tau) d\tau. \quad (3.17)$$

The operator $\hat{\mathcal{J}}$ together with the imbalance current operator $\hat{\mathcal{I}}$,

$$\hat{\mathcal{I}} = \frac{1}{4}(\hat{N}_a - 2\hat{N}_b), \quad (3.18)$$

and the coherent correlation interconversion current operator $\hat{\mathcal{T}}$,

$$\hat{\mathcal{T}} = \frac{1}{4}(\hat{a}^\dagger \hat{a}^\dagger \hat{b} + \hat{b}^\dagger \hat{a} \hat{a}), \quad (3.19)$$

generates the currents algebra

$$[\hat{\mathcal{T}}, \hat{\mathcal{J}}] = +if(\hat{\mathcal{I}}, \hat{N}), \quad [\hat{\mathcal{T}}, \hat{\mathcal{I}}] = -i\hat{\mathcal{J}}, \quad [\hat{\mathcal{J}}, \hat{\mathcal{I}}] = +i\hat{\mathcal{T}},$$

with

$$f(\hat{\mathcal{I}}, \hat{N}) = \frac{3}{2}\hat{\mathcal{I}}^2 + \frac{1}{4}\hat{N}\hat{\mathcal{I}} - \hat{\mathcal{N}}, \quad (3.20)$$

and

$$\hat{\mathcal{N}} = \frac{\hat{N}}{8} \left(\frac{\hat{N}}{4} + 1 \right). \quad (3.21)$$

With the identification $\hat{L}_x \equiv \hat{\mathcal{T}}$, $\hat{L}_y \equiv \hat{\mathcal{J}}$, and $\hat{L}_z \equiv \hat{\mathcal{I}}$ we can write (3.20) as the deformed momentum angular algebra

$$[\hat{L}_k, \hat{L}_l] = i\varepsilon_{klx}\hat{L}_x + i\varepsilon_{kly}\hat{L}_y + i\varepsilon_{klz}f(\hat{L}_z, \hat{N}), \quad (3.22)$$

where ε_{klm} is the antisymmetric Levi-Civita tensor, with $k, l = x, y, z$ and $\varepsilon_{xyz} = +1$.

We have two Casimir operators for that current algebra using the deformed momentum angular realization. One of them is the total number of atoms \hat{N} , related to the global $U(1)$ gauge symmetry, $\hat{\mathfrak{C}}_1 = \hat{N}$, and the another one is related to the deformed momentum angular algebra with the broken $O(3)$ symmetry

$$\hat{\mathfrak{C}}_2 = \hat{L}_x^2 + \hat{L}_y^2 + \hat{L}_z^2 + \frac{\hat{N}}{4}\hat{L}_z^2 + \frac{1}{2} \left(1 - \frac{\hat{N}}{2} - \frac{\hat{N}^2}{8} \right) \hat{L}_z. \quad (3.23)$$

We can show that $\hat{\mathfrak{C}}_2$ is just a function of $\hat{\mathfrak{C}}_1$

$$\hat{\mathfrak{C}}_2 = \frac{\hat{\mathfrak{C}}_1^2}{16} \left(\frac{\hat{\mathfrak{C}}_1}{4} + 1 \right). \quad (3.24)$$

The Casimir operators $\hat{\mathfrak{C}}_1$ and $\hat{\mathfrak{C}}_2$, the boson number of unbound atoms \hat{N}_a , the boson number of molecules \hat{N}_b , and the imbalance current operator, $\hat{\mathcal{I}}$, are CCO and so they have the same set of eigenfunctions and can simultaneous have well defined values

$$\hat{\mathfrak{C}}_2 |n_a, n_b\rangle = \frac{N^2}{16} \left(\frac{N}{4} + 1 \right) |n_a, n_b\rangle, \quad (3.25)$$

$$\hat{\mathcal{I}} |n_a, n_b\rangle = \frac{1}{4} (n_a - 2n_b) |n_a, n_b\rangle. \quad (3.26)$$

We also can use the realization of a deformed $SU(2)$ algebra [61, 62, 69, 70]

$$\hat{\mathcal{L}}_{\pm} = \hat{L}_x \pm i\hat{L}_y, \quad \hat{\mathcal{L}}_z = \hat{L}_z, \quad (3.27)$$

with the commutation relations

$$[\hat{\mathcal{L}}_z, \hat{\mathcal{L}}_{\pm}] = \pm \hat{\mathcal{L}}_{\pm}, \quad [\hat{\mathcal{L}}_+, \hat{\mathcal{L}}_-] = 2f(\hat{\mathcal{L}}_z, \hat{N}),$$

that we can write as

$$[\hat{\mathcal{L}}_k, \hat{\mathcal{L}}_l] = \varepsilon_{kl-} \hat{\mathcal{L}}_+ + \varepsilon_{kl+} \hat{\mathcal{L}}_- + 2\varepsilon_{zkl} f(\hat{\mathcal{L}}_z, \hat{N}), \quad (3.28)$$

with $k, l = z, +, -$ and $\varepsilon_{z+-} = +1$.

This deformed $SU(2)$ algebra has three Casimir operators, $\hat{\mathfrak{C}}_1$,

$$\hat{\mathfrak{C}}_3 = \hat{\mathcal{L}}_+ \hat{\mathcal{L}}_- + \hat{\mathcal{L}}_z^3 + \frac{1}{2} \left(\frac{\hat{N}}{2} - 3 \right) \hat{\mathcal{L}}_z^2 + \frac{1}{2} \left[1 - \hat{N} \left(\frac{\hat{N}}{8} + 1 \right) \right] \hat{\mathcal{L}}_z \quad (3.29)$$

and

$$\hat{\mathfrak{C}}_4 = \hat{\mathcal{L}}_- \hat{\mathcal{L}}_+ + \hat{\mathcal{L}}_z^3 + \frac{1}{2} \left(\frac{\hat{N}}{2} + 3 \right) \hat{\mathcal{L}}_z^2 + \frac{1}{2} \left(1 - \frac{\hat{N}^2}{8} \right) \hat{\mathcal{L}}_z. \quad (3.30)$$

That we can write, using the $\hat{\mathfrak{C}}_1$ Casimir operator, as

$$\hat{\mathfrak{C}}_3 = \frac{1}{64} (\hat{\mathfrak{C}}_1^3 + 2\hat{\mathfrak{C}}_1^2 - 8\hat{\mathfrak{C}}_1), \quad (3.31)$$

$$\hat{\mathfrak{C}}_4 = \frac{1}{64}(\hat{\mathfrak{C}}_1^3 + 6\hat{\mathfrak{C}}_1^2 + 8\hat{\mathfrak{C}}_1). \quad (3.32)$$

They are related by the equation,

$$\hat{\mathfrak{C}}_4 = \hat{\mathfrak{C}}_3 + \frac{\hat{\mathfrak{C}}_1}{4} \left(\frac{\hat{\mathfrak{C}}_1}{4} + 1 \right). \quad (3.33)$$

Using the identities

$$\hat{\mathcal{L}}_+ \hat{\mathcal{L}}_- = \hat{\mathcal{L}}_x^2 + \hat{\mathcal{L}}_y^2 + f(\hat{\mathcal{L}}_z, \hat{N}), \quad \hat{\mathcal{L}}_- \hat{\mathcal{L}}_+ = \hat{\mathcal{L}}_x^2 + \hat{\mathcal{L}}_y^2 - f(\hat{\mathcal{L}}_z, \hat{N}), \quad (3.34)$$

we can write the Cassimir operators (3.29) and (3.30) as

$$\hat{\mathfrak{C}}_3 = \hat{\mathcal{L}}_x^2 + \hat{\mathcal{L}}_y^2 + \hat{\mathcal{L}}_z^3 + \frac{\hat{N}}{4} \hat{\mathcal{L}}_z^2 + \frac{1}{2} \left(1 - \frac{\hat{N}}{2} - \frac{\hat{N}^2}{8} \right) \hat{\mathcal{L}}_z - \hat{\mathcal{N}}, \quad (3.35)$$

$$\hat{\mathfrak{C}}_4 = \hat{\mathcal{L}}_x^2 + \hat{\mathcal{L}}_y^2 + \hat{\mathcal{L}}_z^3 + \frac{\hat{N}}{4} \hat{\mathcal{L}}_z^2 + \frac{1}{2} \left(1 - \frac{\hat{N}}{2} - \frac{\hat{N}^2}{8} \right) \hat{\mathcal{L}}_z + \hat{\mathcal{N}}. \quad (3.36)$$

It is easy to show that $\hat{\mathfrak{C}}_4 - \hat{\mathcal{N}} = \hat{\mathfrak{C}}_3 + \hat{\mathcal{N}} = \hat{\mathfrak{C}}_2$ and $\hat{\mathfrak{C}}_4 - \hat{\mathfrak{C}}_3 = 2\hat{\mathcal{N}}$. Therefore, if we take the average value as in [61, 62, 70] we see that the Casimir operators $\hat{\mathfrak{C}}_2$, $\hat{\mathfrak{C}}_3 + \hat{\mathcal{N}}$ and $\hat{\mathfrak{C}}_4 - \hat{\mathcal{N}}$ describe the same surfaces. Some of these Casimir surfaces has been denoted as Kummer shapes [61–68]. We plot these surfaces in the Fig. (1) using the following parametrization

$$X(u, v) = f(v) \cos u, \quad (3.37)$$

$$Y(u, v) = f(v) \sin u, \quad (3.38)$$

$$Z(u, v) = b v, \quad (3.39)$$

where

$$f(v) = \sqrt{C_2 - v^3 - \frac{N}{4}v^2 - \frac{1}{2} \left(1 - \frac{N}{2} - \frac{N^2}{8} \right) v}, \quad (3.40)$$

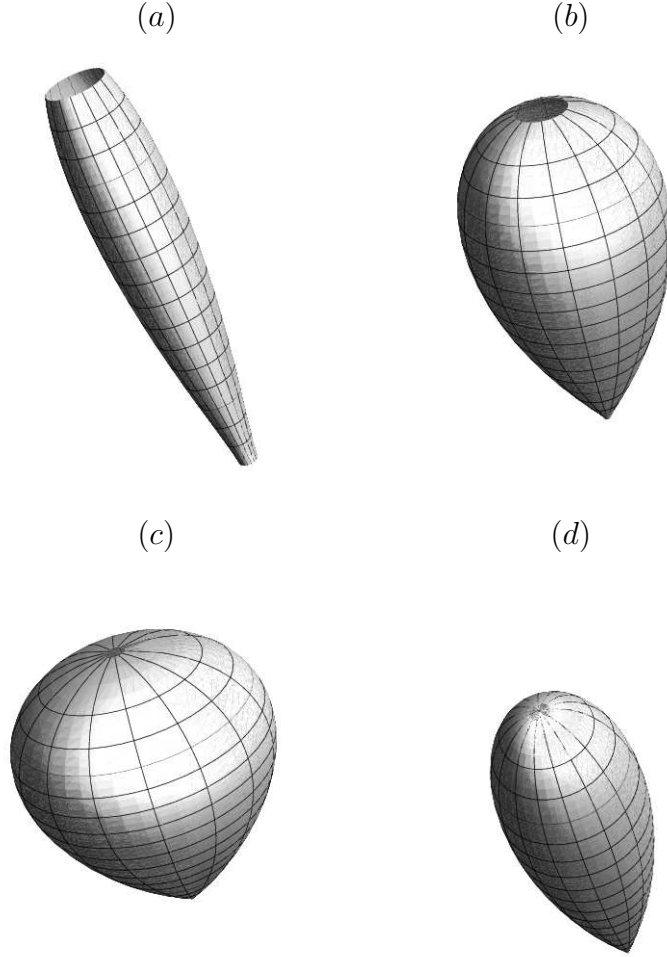


Figure 1: Casimir surfaces for (a) $N = 10$ and $b = 10$, (b) $N = 100$ and $b = 10$, (c) $N = 1000$ and $b = 20$, (d) $N = 15000$ and $b = 200$.

with $C_2 = \frac{N^2}{16} \left(\frac{N}{4} + 1 \right)$ the eigenvalue of the Casimir operator $\hat{\mathfrak{C}}_2$, $0 \leq u \leq 2\pi$ and $-\frac{N}{4} \leq v \leq \frac{N}{4}$. The parameter b changes the scale in the Z -axis direction. The radii in the boundaries of the Z -axis direction are $f(-\frac{N}{4}) = \sqrt{\frac{N}{8}}$ and $f(\frac{N}{4}) = \sqrt{\frac{N}{8}(N-1)}$. If we extend the physical limit of v we get the surfaces in Fig. (2).

Using the commutation relations of the currents (3.20) it is easy to calculate the anticommutators

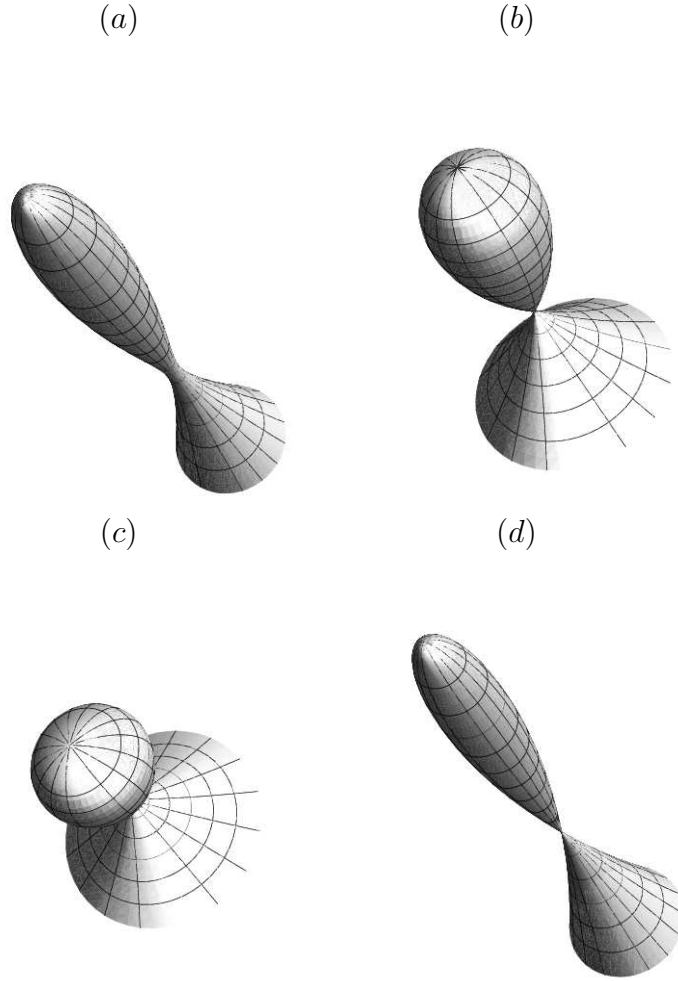


Figure 2: Casimir surfaces for $v \in [-0.55N, 0.55N]$ and (a) $N = 10$ and $b = 10$, (b) $N = 100$ and $b = 10$, (c) $N = 1000$ and $b = 20$, (d) $N = 15000$ and $b = 100$.

$$[\hat{\mathcal{T}}, \hat{\mathcal{I}}]_+ = 2\hat{\mathcal{I}}\hat{\mathcal{T}} - i\hat{\mathcal{J}}, \quad (3.41)$$

$$[\hat{\mathcal{T}}, \hat{\mathcal{J}}]_+ = 2\hat{\mathcal{J}}\hat{\mathcal{T}} + if(\hat{\mathcal{I}}, \hat{N}), \quad (3.42)$$

$$[\hat{\mathcal{J}}, \hat{\mathcal{I}}]_+ = 2\hat{\mathcal{I}}\hat{\mathcal{J}} + i\hat{\mathcal{T}}. \quad (3.43)$$

We will use these anticommutators together with the commutators (3.20) in the calculus of the currents quantum dynamics.

4 Interconversion Currents Quantum Dynamics

We can rewrite the Hamiltonian (1.1) using the currents operators (3.12), (3.18) and (3.19) as

$$\hat{H} = \alpha\hat{\mathcal{I}}^2 + \hat{\zeta}\hat{\mathcal{I}} - 4\Omega\hat{\mathcal{T}} + \rho\hat{\mathfrak{C}}_1^2 + \xi\hat{\mathfrak{C}}_1, \quad (4.44)$$

with the following identification for the parameters

$$\begin{aligned} \alpha &= 4U_a + U_b - 2U_{ab}, \\ \beta &= \left(2U_a - \frac{U_b}{2}\right), \\ \gamma &= 2\mu_a - \mu_b, \\ \rho &= \left(\frac{U_a}{4} + \frac{U_b}{16} + \frac{U_{ab}}{8}\right), \\ \xi &= \left(\frac{\mu_a}{2} + \frac{\mu_b}{4}\right), \end{aligned}$$

and with the definition of the Casimir operator $\hat{\zeta} = \beta\hat{\mathfrak{C}}_1 + \gamma$.

The quantum dynamic of the currents are determined by the currents algebra (3.20), their commutation relations with the Hamiltonian and the parameters. Following [49] we can write the second time derivative of any operator \hat{O} in the Heisenberg picture as

$$\frac{d^2\hat{O}}{dt^2} = \left(\frac{i}{\hbar}\right)^2 [\hat{H}, [\hat{H}, \hat{O}]], \quad (4.45)$$

or as

$$\frac{d^2\hat{O}}{dt^2} = \frac{i}{\hbar} [\hat{H}, \frac{d\hat{O}}{dt}]. \quad (4.46)$$

We can see from the Hamiltonian (4.44) and Eq. (4.45) that all Casimir operators are conserved quantities, $[\hat{H}, \hat{\mathfrak{C}}_1] = 0$. Because the operator \hat{N} is a Casimir operator and a

conserved quantity all wave function of the Hamiltonians (1.1) or (4.44) are eigenfunctions of that operator with degenerated eigenvalue N [45]. As the another Casimir operators are functions of \hat{N} we have also that all wave function of the Hamiltonians are degenerated eigenfunctions of these operators with the respective eigenvalues as functions of N . Therefore, we will consider the Casimir operators $\hat{\zeta}$ and \hat{N} as the respective c -numbers N and $\zeta = \beta N + \gamma$ in the calculus of the currents quantum dynamics. So we also will write $\hat{\mathcal{N}}$ and $f(\hat{\mathcal{I}}, \hat{N})$ as the respective functions of N ,

$$\mathcal{N} = \frac{N}{8} \left(\frac{N}{4} + 1 \right), \quad f(\hat{\mathcal{I}}, N) = \frac{3}{2} \hat{\mathcal{I}}^2 + \frac{N}{4} \hat{\mathcal{I}} - \mathcal{N}. \quad (4.47)$$

Using the Eq. (4.44) and (4.45) or (4.46) we found the following equations for the quantum dynamics of the three currents

$$\frac{d^2 \hat{\mathcal{I}}}{dt^2} + i \frac{\alpha}{\hbar} \frac{d \hat{\mathcal{I}}}{dt} + 24 \frac{\Omega^2}{\hbar^2} \hat{\mathcal{I}}^2 + 4 \frac{\Omega^2}{\hbar^2} N \hat{\mathcal{I}} = -8 \frac{\Omega \alpha}{\hbar^2} \hat{\mathcal{I}} \hat{\mathcal{T}} - 4 \frac{\Omega \zeta}{\hbar^2} \hat{\mathcal{T}} + 16 \frac{\Omega^2}{\hbar^2} \mathcal{N}, \quad (4.48)$$

$$\begin{aligned} \frac{d^2 \hat{\mathcal{J}}}{dt^2} + \frac{1}{\hbar^2} [\alpha^2 + 4\Omega^2 N + \zeta^2] \hat{\mathcal{J}} &= -4 \frac{\alpha^2}{\hbar^2} \hat{\mathcal{I}}^2 \hat{\mathcal{J}} - 4i \frac{\alpha^2}{\hbar^2} \hat{\mathcal{I}} \hat{\mathcal{T}} - 8 \frac{\Omega \alpha}{\hbar^2} \hat{\mathcal{J}} \hat{\mathcal{T}} \\ &- \frac{4}{\hbar^2} (\alpha \zeta + 12\Omega^2) \hat{\mathcal{I}} \hat{\mathcal{J}} - \frac{2i}{\hbar^2} (\alpha \zeta + 12\Omega^2) \hat{\mathcal{T}} \\ &- 6i \frac{\Omega \alpha}{\hbar^2} \hat{\mathcal{I}}^2 - i \frac{\Omega \alpha}{\hbar^2} N \hat{\mathcal{I}} + 4i \frac{\Omega \alpha}{\hbar^2} \mathcal{N}, \end{aligned} \quad (4.49)$$

$$\begin{aligned} \frac{d^2 \hat{\mathcal{T}}}{dt^2} + \frac{1}{\hbar^2} [\alpha^2 + \zeta^2] \hat{\mathcal{T}} &= -4 \frac{\alpha^2}{\hbar^2} \hat{\mathcal{I}}^2 \hat{\mathcal{T}} + 4i \frac{\alpha^2}{\hbar^2} \hat{\mathcal{I}} \hat{\mathcal{J}} - 4 \frac{\alpha \zeta}{\hbar^2} \hat{\mathcal{I}} \hat{\mathcal{T}} + 2i \frac{\alpha \zeta}{\hbar^2} \hat{\mathcal{J}} \\ &+ 8 \frac{\Omega \alpha}{\hbar^2} \hat{\mathcal{J}}^2 - 8 \frac{\Omega \alpha}{\hbar^2} \hat{\mathcal{I}} f(\hat{\mathcal{I}}, N) - 4 \frac{\Omega \zeta}{\hbar^2} f(\hat{\mathcal{I}}, N). \end{aligned} \quad (4.50)$$

Different choices of the parameters gives us different dynamics for the currents. The parameters ρ and ξ don't change the dynamics of the currents because they are coupling the Cassimir operator $\hat{\mathcal{C}}_1$. For $\alpha = \zeta = 0$, the current $\hat{\mathcal{T}}$ is a conserved quantity, $[\hat{\mathcal{H}}, \hat{\mathcal{T}}] = 0$, but this don't means that we don't have interconversion dynamics. We can see from Eqs. (3.16) and (3.17) that the quantum dynamic of \hat{N}_a , \hat{N}_b , and $\hat{\mathcal{I}}$ only depend of the interconversion

current $\hat{\mathcal{J}}$ and the interconversion parameter Ω . We only have dynamics if $\Omega \neq 0$. We get the following relations between the interactions and external potentials parameters

$$U_{ab} = 2U_a + \frac{U_b}{2}, \quad \frac{\mu_b - 2\mu_a}{N} = 2U_a - \frac{U_b}{2}. \quad (4.51)$$

The simplest choice is the non interaction limit with all U 's equal to zero and with all external potentials μ 's equal to zero, but the equations for the currents quantum dynamics are the same since that the parameters obey the relations (4.51). For $\alpha = 0$ and $\zeta \neq 0$, we have yet the external potentials μ 's and the atom-atom U_a , molecule-molecule U_b and atom-molecule U_{ab} interactions, but the interactions are constrained by the first equation in (4.51). We can also choose $2\mu_a = \mu_b$ or $U_a = U_b/4$ and we will get $\zeta \neq 0$.

5 Summary

I have discussed the symmetries of the model and calculated the Casimir operators for the $O(3)$ and $SU(2)$ deformed algebras and showed that the surfaces corresponding to these Casimir operators are the same. I have showed that an interconversion currents algebra appears when I calculate the quantum dynamics of the unbound atoms \hat{N}_a and molecules \hat{N}_b boson number operators. I have used the Heisenberg equation of motion to write the second time derivative of the currents operators. Then I calculated the quantum dynamics of these currents and showed that different dynamics appear when I consider different choices of the parameters of the Hamiltonian.

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